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LETTER TO THE EDITOR

States of minimal joint uncertainty for complementary observables in three-dimensional Hilbert space

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Abstract. The entropic uncertainty relation for sets of $N+1$ complementary observables $\{A_r\}$ existing in N-dimensional Hilbert space, $\sum_r H(A_r) \geqslant (N+1)\ln((N+1)/2)$, is shown to be optimal in the case $N=3$ by explicit construction of the states for which equality holds. We prove that the lower bound cannot be attained when N is even, and, on the basis of numerical calculations, this is conjectured to also be the case for odd $N>3$.

Two observables A and B of a quantum system with N-dimensional Hilbert space are said to be complementary, or mutually unbiased, if their eigenvalues are non-degenerate, and inner products between any two normalized eigenvectors $|a_i\rangle$ of A and $|b_j\rangle$ of B all have the same magnitude [1, 2],

$$|\langle a_i \mid b_j\rangle| = \frac{1}{\sqrt{N}} \qquad (i, j = 1, \ldots, N). \tag{1}$$

For any (normalized) state vector $|\psi\rangle$, the probability distributions $\{p_i(A)\}$ and $\{p_i(B)\}$, given by

$$p_i(A) = |\langle\Psi \mid a_i\rangle|^2 \qquad p_i(B) = |\langle\Psi \mid b_i\rangle|^2 \tag{2}$$

satisfy the inequality

$$H(A) + H(B) \geqslant \ln N \tag{3}$$

where $H(A)$ and $H(B)$ are their corresponding information entropies,

$$H(A) = -\sum_{i=1}^{N} p_i(A)\ln p_i(A) \qquad H(B) = -\sum_{i=1}^{N} p_i(B)\ln p_i(B). \tag{4}$$

As conjectured by Kraus [2] and proved by Maassen and Uffink [3], inequality (3) is the optimal entropic uncertainty relation for the pair (A, B), the lower bound being attained for eigenstates of any of these two observables.

Pairs of complementary observables exist for arbitrary values of N [2, 4]. Furthermore, sets $\{A_r\}$ of $N+1$ pairwise complementary observables can be found when N is a prime

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number [5] and, more generally, whenever N is a power of a prime [6]. For such a set, a stronger entropic uncertainty relation has recently been established [7, 8],

$$\sum_{r=1}^{N+1} H(A_r) \geqslant (N+1)\ln\left(\frac{N+1}{2}\right). \tag{5}$$

Here we discuss the conditions under which (5) holds with equality, showing that they cannot be satisfied when N is even. We also prove that (5) is optimal for $N = 3$ by explicitly constructing the set of states that achieve the lower bound in the right-hand side, whereas numerical calculations suggest that (5) is not optimal for odd $N > 3$.

Inequality (5) can be derived from the following result, first obtained by Larsen [9]

$$\sum_{r=1}^{N+1} \pi(A_r) = 1 + \mathrm{Tr}(\rho^2) \leqslant 2 \tag{6}$$

where ρ is the density matrix describing a general (pure or mixed) quantum state, and

$$\pi(A_r) = \sum_{i=1}^{N} [p_i(A_r)]^2. \tag{7}$$

We have [7, 8]

$$\sum_{r=1}^{N+1} H(A_r) \geqslant -\sum_{r=1}^{N+1} \ln \pi(A_r) \geqslant -(N+1)\ln\left(\frac{\sum_{r=1}^{N+1}\pi(A_r)}{N+1}\right)$$

$$\geqslant -(N+1)\ln\left(\frac{2}{N+1}\right) = (N+1)\ln\left(\frac{N+1}{2}\right).$$

The last inequality in this chain, following from (6), is satisfied with equality for pure states ($\mathrm{Tr}(\rho^2) = 1$). The inequality in the middle follows from the concavity of the logarithm and becomes an equality iff the positive quantities $\pi(A_r)$ all have the same value,

$$\pi(A_r) = \frac{2}{N+1} \qquad (r = 1, \ldots, N+1) \tag{8}$$

as (6) implies for pure states. On the other hand, $H(A_r) \geqslant -\ln \pi(A_r)$ is the special case $s = 0$, $t = 1$ of a more general inequality,

$$\left(\sum_i [p_i(A_r)]^{s+1}\right)^{1/s} \leqslant \left(\sum_i [p_i(A_r)]^{t+1}\right)^{1/t} \qquad s < t$$

where equality holds iff the distribution $\{p_i(A_r)\}$ is such that all non-zero probabilities are equal [10]. On the assumption that there are N_r such probabilities ($1 \leqslant N_r \leqslant N$), with common value $p_r = 1/N_r$, from (7) and (8) we get

$$N_r = \frac{N+1}{2} \qquad (r = 1, \ldots, N+1). \tag{9}$$

Now it is obvious that this condition cannot be fulfilled when N is even, so that the entropic uncertainty relation (5) is then not optimal. It is worth noting that in the simplest case $N = 2$, where complementary observables are three mutually orthogonal spin-$\frac{1}{2}$ components, a direct calculation shows that the optimal lower bound on the entropy sum is not $3\ln(3/2)$, as given by (5), but $2\ln 2$, which is attained for eigenstates of these complementary observables [8].

When N is odd, condition (9) would be satisfied for (pure) states such that

$$\begin{aligned} p_i(A_r) &= |\langle \Psi \mid a_i^{(r)} \rangle|^2 \\ &= \left\{ \frac{2}{N+1} \left(i = 1, \ldots, \frac{N+1}{2} \right),\ 0 \text{ otherwise} \right\} \qquad (r = 1, \ldots, N+1) \end{aligned} \tag{10}$$

(up to rearrangements). Therefore, the lower bound in (5) is only attained for states whose representation in every basis $\{|a_i^{(r)}\rangle\} (r = 1, \ldots, N+1)$ is a column matrix of the form

$$\sqrt{\frac{2}{N+1}} \begin{pmatrix} e^{i\varphi_1} \\ e^{i\varphi_2} \\ \vdots \\ e^{i\varphi_{\frac{N-1}{2}}} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{11}$$

up to rearrangements of the elements and overall phases. Next we shall prove by explicit construction that such states do actually exist in the case $N = 3$.

For N prime, we have [5, 6]

$$\langle a_k^{(r)} \mid \Psi \rangle = \sum_{l=0}^{N-1} \langle a_k^{(r)} \mid a_l^{(N+1)} \rangle \langle a_l^{(N+1)} \mid \Psi \rangle \tag{12}$$

where

$$\langle a_k^{(r)} \mid a_l^{(N+1)} \rangle = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} (rl^2 + kl) \right) \qquad (k, l = 0, 1, \ldots, N-1; r = 1, \ldots, N). \tag{13}$$

In particular, for $N = 3$, these transformation matrices are

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \alpha & \alpha \\ 1 & \bar{\alpha} & 1 \\ 1 & 1 & \bar{\alpha} \end{pmatrix} \qquad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \bar{\alpha} & \bar{\alpha} \\ 1 & 1 & \alpha \\ 1 & \alpha & 1 \end{pmatrix} \qquad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \bar{\alpha} \\ 1 & \bar{\alpha} & \alpha \end{pmatrix} \tag{14}$$

for $r = 1, 2, 3$ respectively, where $\alpha = \exp(2\pi i/3)$ and $\bar{\alpha}$ denotes complex conjugate of α. Multiplication by a column matrix of the form (11),

$$\frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi} \\ 1 \\ 0 \end{pmatrix} \tag{15}$$

then gives a rearrangement of

$$\frac{1}{\sqrt{6}}\begin{pmatrix}\alpha+\mathrm{e}^{\mathrm{i}\varphi}\\ \bar{\alpha}+\mathrm{e}^{\mathrm{i}\varphi}\\ 1+\mathrm{e}^{\mathrm{i}\varphi}\end{pmatrix}$$

which is also of the form (15) if and only if $\varphi=\pi, \pm\pi/3$. Thus we conclude that inequality (5) is optimal when $N=3$, and the lower bound $4\ln 2 = 2.772\,59$ is attained for

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}\qquad \frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\pi/3}\\ 1\\ 0\end{pmatrix}\qquad \frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\pi/3}\\ 1\\ 0\end{pmatrix} \tag{16}$$

which are states of minimal joint (information-theoretic) uncertainty for a set of complementary observables in three-dimensional Hilbert space.

For $N \geqslant 5$, the column matrices $\{\langle\psi \mid a_i^{(r)}\rangle\}$ obtained from a $\{\langle\psi \mid a_i^{(N+1)}\rangle\}$ of the form (11) by using (12) and (13) are no longer rearrangements of the same elements, so we may expect that they cannot be simultaneously of the form (11), and then it seems reasonable to conjecture that (5) is only optimal when $N=3$. For example, numerical calculations reveal that in the case $N=5$ the minimum value of the entropy sum for states with a representation of the form (11) is 7.493 59, while the states

$$\frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\varphi}\\ 1\\ 0\\ 0\\ 0\end{pmatrix}\qquad \varphi=\pi, \pm\frac{\pi}{5}, \pm\frac{3\pi}{5} \tag{17}$$

which are strikingly similar to the minimum uncertainty states of the three-dimensional case, yield for the entropy sum a lesser value, namely

$$6\ln 2+\frac{5}{2}\ln 5-\frac{\sqrt{5}}{2}\ln\left(\frac{3+\sqrt{5}}{2}\right)=7.106\,46 \tag{18}$$

which is still significantly higher than the bound given by (5), $6\ln 3 = 6.591\,67$. We conjecture that these are states of minimal joint uncertainty in five-dimensional Hilbert space, but a proof (or counter-example) is still lacking. The general problem of finding the optimal lower bound on the entropy sum for complementary observables and the corresponding states of minimal joint uncertainty for $N>3$ is a matter of future research.

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